INDETERMINATE CASES IN THE ORBIT PROBLEM

A Ph. D. thesis

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PURPOSE.

In general the problem of determining the orbit of a minor planet or comet from three observations is a definite and solvable problem. But there are certain cases which, altogether apart from the indeterminateness due to uncertainties of the observations, do not yield results to the ordinary methods of attack. These indeterminate cases have been discussed in greater or less detail by many writers on the orbit problem. It is the purpose of this paper to review the available material on the subject, to ascertain wherein the conclusions reached by different writers are consistent, wherein they differ and to decide if possible whether the indeterminateness that has been found is due to the methods employed or is inherent in the particular geometrical and physical conditions of the case.

HISTORY.

In a letter to Olbers in January 1806 Gauss speaks of one instance in which Olbers' method fails. Using this method with three observations of the comet of 1772 Gauss had obtained a very different orbit from that given by other observations. This, he points out, was due to the fact that the observations he selected lay nearly on a great circle passing through the middle position of the sun. He states, however, that by a transformation of the fundamental formulae he had devised a means whereby a first approximation could be

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obtained in this case. Under these same circumstances his method for planetary orbits is also inapplicable. However, he concludes that there is a fundamental distinction between these two cases. Whereas in the elliptic orbit the problem is "natura sua" indeterminate the parabolic case admits of a solution, though not by the usual method.

Later in his "Theoria Motus" published in 1809, Gauss speaks again of the case where the apparent path of the planet passes through the middle position of the sun as being absolutely indeterminate for elliptic orbits. Another absolutely indeterminate case is the one where the first and third geocentric places coincide. As distinguished from these two he speaks of one case where he believed the indeterminateness was due to the method alone and therefore a result could be obtained by a suitable transformation. This is the case where any one of the three geocentric places coincides either with the corresponding heliocentric place of the earth or with the opposite point.

He shows that two apparently indeterminate cases can be avoided by proper manipulation of the formulae. These are the cases where the first and third geocentric places lie on the same great circle as either the first or the third heliocentric position of the earth.

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In his paper "Uber die Bestimmung der Bahn eines Himmelkorper" published in 1863 Hansen has an interesting discussion concerning the great circle case. He develops the

equation
$$\rho_2' = \frac{\theta\theta''C}{2K}\left[\frac{1}{r_2^3} - \frac{1}{R_2^3}\right]$$
where ρ_2' is the projected geocentric distance at the middle date

$$C = R_2\left[tg\,\beta_1 \sin(\alpha_3 - l_2) - tg\,\beta_3 \sin(\alpha_1 - l_2)\right]$$
$$K = tg\,\beta_1 \sin(\alpha_3 - \alpha_2) - tg\,\beta_2 \sin(\alpha_3 - \alpha_1) + tg\,\beta_3 \sin(\alpha_2 - \alpha_1)$$

Hansen states that from this equation it is apparent, since ρ_2' must be a positive finite quantity, that C and K must have opposite signs if $r_2 > R_2$ and the same signs if $r_2 < R_2$. It shows further that the condition $K = 0$ carries with it the conclusion $C = 0$, except for the case $r_2 = R_2$. When $K = 0$ the three consecutive infinitesimal elements of the geocentric path lie in a great circle and when $C = 0$ the corresponding position of the sun lies on this same great circle.

He continues: If now we go a step further and take the time intervals as still small but finite quantities the state of affairs changes. The equation above has been derived by neglecting quantities of the sixth order for unequal time intervals, the seventh for equal. This is not always possible and when the equation no longer holds we can no longer draw the same conclusions regarding the relationships between C and K. Only this much is sure - with very small time intervals a very small value of K, or the value $K = 0$, will imply in general a small value of C, but if the time intervals are not small C can even reach its maximum while K is very small or even zero. It is then necessary to return to the original equations which do not neglect the small quantities mentioned above.

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From the original equation Hansen proves that in the indeterminate case where the three observations lay on a great circle passing through the middle position of the sun the ratios of the areas of the triangles swept over by the planet and the earth are the same. This is shown as follows from the equation:

$$\left\{A - C + P(B - C) - K(P+1)\rho_2' \cos\beta\right\} r_2^3 + \tfrac{1}{2} Q (A + PB) = 0$$

where K and C have the values given above

$$A = R_1 \left\{ tg\beta_1 \sin(\alpha_3 - l_1) - tg\beta_3 \sin(\alpha_1 - l_1) \right\}$$

$$B = R_3 \left\{ tg\beta_1 \sin(\alpha_3 - l_3) - tg\beta_3 \sin(\alpha_1 - l_3) \right\}$$

$$P = \frac{n''}{n} \qquad Q = 2r_2^3\left(\frac{n+n''}{n'} - 1\right)$$

When K = C = 0 we have

$$(A + BP)\left(r_2^3 + \tfrac{1}{2}Q\right) = 0$$

Since the second factor cannot vanish, the first must or:

$$P = -\frac{A}{B} = \frac{R_1}{R_3}\left\{\frac{tg\beta_1 \sin(\alpha_3 - l_1) - tg\beta_3 \sin(\alpha_1 - l_1)}{tg\beta_1 \sin(\alpha_3 - l_3) - tg\beta_3 \sin(\alpha_1 - l_3)}\right\}$$

If now we eliminate the quantities $tg\beta_1$ and $tg\beta_3$ by means of the equations

$$tg\beta_1 = tg\eta \sin(\alpha_1 - l_2)$$

$$tg\beta_3 = tg\eta \sin(\alpha_3 - l_2)$$

where η is the angle which the great circle of apparent motion makes with the ecliptic we have:

$$P = \frac{R_1}{R_3}\left\{\frac{\sin(\alpha_1 - l_2)\sin(\alpha_3 - l_1) - \sin(\alpha_3 - l_2)\sin(\alpha_1 - l_2)}{\sin(\alpha_3 - l_2)\sin(\alpha_1 - l_3) - \sin(\alpha_1 - l_2)\sin(\alpha_3 - l_1)}\right\}$$

This reduces to the form

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But we already had $P = \frac{n''}{n} \therefore \frac{n''}{n} = \frac{\eta''}{\eta}$ or the ratios of the triangles are identical.

The equation for ρ_2 reduces to the identity $0 = 0$ in the case where the two outer positions coincide. In regard to the case where the middle position is in opposition or conjunction and also in the ecliptic Hansen states that the problem can be solved but the value of r_2 is given by the quotient of terms of so high an order that it will be greatly affected by the errors of observation. This is also true, though to a less degree, if either the first or third observation fulfills the same conditions.

Oppolzer puts a different emphasis on the conclusion in regard to great circle motion. Since r can never equal R in the case of the minor planets and since either this condition must hold or the condition that the sun lies on the same great circle, which affords an indeterminate case, he draws the conclusion that in general the determination of the orbit problem becomes impossible when the three positions lie in the same great circle and is very uncertain when this is nearly the case. It would then be necessary to take four observations.

It would be well to point out at this time that Oppolzer's conclusion is certainly not valid in the case of comets as there great circle motion is fully determinate.

Watson in his Theoretical Astronomy (1868) gives little that is new. The same two indeterminate cases are discussed briefly. He shows how several cases which might at first appear indeterminate can be avoided by a suitable

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transformation of equations. Among these cases are the following:

(1) When the planet or comet at the time of the middle observation is near both its node and opposition or conjunction we first obtain r_2 by a method of approximation from the equation: $k_0 - \frac{l_0}{r_2^3} = -R_2 + r_2$

where $$k_0 = \frac{\cos\beta_0 \, tg\, I}{\sin(\beta_2 - \beta_0)}\left\{\frac{R_1 \sin(\odot_1 - K) + P R_3 \sin(\odot_3 - K)}{1+P} - R_2 \sin(\odot_2 - K)\right\}$$

$$l_0 = -\frac{1}{2}\,\frac{Q \cos\beta_0 \, tg\, I}{\sin(\beta_2 - \beta_0}\left\{\frac{R_1 \sin(\odot_1 - K) + P R_3 \sin(\odot_3 - K)}{1+P}\right\}$$

in the first approximation $P = \frac{\theta_3}{\theta_1}$ $Q = \theta_1 \theta_3$

K is the longitude of the ascending node and I the inclination to the ecliptic of the great circle passing through the first and third observed places of the body. β_0 is given by the equation: $tg\,\beta_0 = tg\, I \sin(\lambda_2 - K)$

After r_2 is found ρ_2 is given by either

$$\rho_2 = r_2 - R_2 \text{ or } \rho_2 = R_2 - r_2 .$$

(2) When $\lambda_1 = \lambda_3$ but $\beta_1 \mp \beta_3$

(3) When the great circle passing through the first and third observed places of the body also passes through the first or third place of the sun.

In the case of Olbers' method for parabolic orbits he gives an alternative formula for determining ρ_2 to be used in the exceptional case; i.e. when the three observations lie on the same great circle passing through the middle position of the sun.

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where $h_0 = \frac{\cos\beta_0 \, tg\, I}{\sin(\beta_2-\beta_0)}\left\{\frac{R_1 \sin(\Theta_1-K) + T R_3 \sin(\Theta_3-K)}{1+T} - R_2 \sin(\Theta_2-K)\right\}$

$l_0 = \frac{1}{2}\,\frac{g \cos\beta_0 \, tg\, I}{\sin(\beta_2-\beta_0)}\left\{\frac{R_1 \sin(\Theta_1-K) + T R_3 \sin(\Theta_3-K)}{1+T}\right\}$

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In 1869 Tietjen published a paper entitled "Uber die Unsicherheit einer Bahnbestimmung aus drei Beobachtungen, wenn dieselben geocentrisch nahe in einem grossten Kreise liegen". His conclusions concerning the alternative conditions which accompany great circle motion are practically identical with those of Hansen. He discusses in some detail the second; i.e. where K = 1/3 $(\odot_1 + \odot_2 + \odot_3)$

K = longitude of ascending node of great circle through geocentric positions.

$\odot_1 \odot_2 \odot_3$ = longitudes of sun on the three dates.

The following relation is shown to hold at such a time.

$$\frac{r_0}{\sqrt{p}} \sin u_0 = - R_0 \sin (\odot_0 - \Omega)$$

where p is parameter of orbit

u is argument of latitude.

Further he gives an example, the planet Phocaea which fulfilled these conditions near May 8.6, 1853.

A study of the equation above shows it must hold when the body is near both opposition and one of its nodes. If $r_0 > R_0$ it cannot hold for u near 90° and 270° since then $\sin(\odot_0 - \Omega)$ would become greater than unity. In general it will hold for small values of u not far from opposition.

Klinkerfues-Buchholtz: An interesting discussion is given concerning the loop case in the geocentric motion of a planet or comet. On the surface this appears indeterminate since the position of the great circle through the two outer places is indeterminate. He proves the following theorem: A great circle joining any intermediate point on the loop to

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the corresponding position of the sun also passes through the double point, at least very approximately. After a rigorous proof he points out that the same conclusion could have been deduced from Lambert's principle. We know that if a great circle through the first and third geocentric places divides the celestial sphere into two parts, the middle position will lie on the same half sphere as the sun if its distance from the sun exceeds that of the earth and on the opposite half sphere if its distance is less. In this particular case the position of the great circle is arbitrary and we could apparently derive conflicting results by letting the sun and middle planet position lie either on the same or on opposite half spheres or allowing the three positions to lie on the same great circle. However Lambert's principle may give an indeterminate result but never a false one. The indeterminate result is indicated by the fact that the three positions and the sun all lie on the same great circle and this is what must hold in this instance. An approximate value of r_2 can be found.

This theorem shows that if we wish we could consider the loop case as a special instance of motion in a great circle through the middle place of the sun.

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minate because its determination could be made independent of r_2 through transformations". He dismisses it with the remark that in general the case can be avoided by a proper choice of the observations.

He considers great circle motion under two headings.

(1) The great circle does not pass through the middle position of the sun. Here $r_2 = R_2$ and the problem is easily solved.

(2) The sun lies on the same great circle as the three geocentric positions. He brings out the fact that since $\frac{m_1}{m_3} = \frac{n_1}{n_3}$ if the time intervals are unequal $r_2 = R_2$. This is evident from the expansions of the ratios of the triangles in terms of the time, if it is possible to neglect terms of the higher order.

$$\frac{m_1}{m_3} = \frac{\theta_1}{\theta_3}\left\{1 + \frac{\theta_1^2 - \theta_3^2}{6 r_2^3} - \frac{\theta_1^3 + \theta_3^3}{4 r_2^4}\frac{dr_2}{dt} + \dots\right\}$$

$$\frac{n_1}{n_3} = \frac{\theta_1}{\theta_3}\left\{1 + \frac{\theta_1^2 - \theta_3^2}{6 R_2^3} - \frac{\theta_1^3 + \theta_3^3}{4 R_2^4}\frac{dR_2}{dt} + \dots\right\}$$

If now $\theta_1 \neq \theta_3$ and it is possible to neglect terms of the third and higher order r_2 must equal R_2.

If the time intervals however are even the orbit remains indeterminate. Weiss makes the statement that when $r_2 \neq R_2$ the motion does not lie in a great circle but in a curve which has a point of inflexion at the middle place and that this point lies in the line connecting the first and third observations. This can evidently occur only under rare circumstances.

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His conclusion is that the body can only and must move only in a great circle when $r_2 = R_2$ nearly. When it is impossible to determine the elements from three observations without restriction as to the excentricity it is still possible to determine the ratios of the geocentric distances with considerable accuracy.

In another article he discusses Olbers' parabolic method and determines the most exact value possible of M (ratio of ρ_1 to ρ_3) in the exceptional case.

In paragraph 100 of his "Bahnbestimmung" Bauschinger discusses the "Ausnahmefalle" with much the same conclusions as Gauss. In the loop case he speaks of the fact that while ρ_2 can be determined, ρ_1 and ρ_3 only appear in the combination $n_1\rho_1 + n_3\rho_3$. This is apparent for if the values $\alpha_1 = \alpha_3$, $\beta_1 = \beta_3$ are substituted in the equations

$$n_1 x_1 - n_2 x_2 + n_3 x_3 = 0$$
$$n_1 y_1 - n_2 y_2 + n_3 y_3 = 0$$
$$n_1 z_1 - n_2 z_2 + n_3 z_3 = 0$$

they become:

$$(n_1\rho_1 + n_3\rho_3)\cos(\alpha_1 - \varepsilon) - n_2\rho_2\cos(\alpha_2 - \varepsilon) + n_1 R_1 \cos l_1 - n_2 R_2 \cos l_2 - n_3$$
$$(n_1\rho_1 + n_3\rho_3)\sin(\alpha_1 - \varepsilon) - n_2\rho_2\sin(\alpha_2 - \varepsilon) + n_1 R_1 \sin l_1 - n_2 R_2 \sin l_2 - n_3 R_3 \sin l_3 =$$
$$(n_1\rho_1 + n_3\rho_3)\,tg\,\beta_1 - n_2\rho_2\,tg\,\beta_2 = 0$$

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Plummer is his Dynamical Astronomy shows that when the determinant $\begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix}$ vanishes (a,b,c,a',b',c',a",b",c"

His conclusion is that the body can only and must move only in a great circle when $r_2 = R_2$ nearly. When it is impossible to determine the elements from three observations without restriction as to the eccentricity it is still possible to determine the ratios of the geocentric distances with considerable accuracy.

In another article he discusses Olbers' parabolic method and determines the most exact value possible of M (ratio of ρ_1 to ρ_3) in the exceptional case.

In paragraph 100 of his "Bahnbestimmung" Bauschinger discusses the "Ausnahmefalle" with much the same conclusions as Gauss. In the loop case he speaks of the fact that while ρ_2 can be determined, ρ_1 and ρ_3 only appear in the combination $m_1\rho_1 + m_3\rho_3$. This is apparent for if the values $\alpha_1 = \alpha_3$, $\beta_1 = \beta_3$ are substituted in the equations

$$m_1 x_1 - m_2 x_2 + m_3 x_3 = 0$$
$$m_1 y_1 - m_2 y_2 + m_3 y_3 = 0$$
$$m_1 z_1 - m_2 z_2 + m_3 z_3 = 0$$

they become:

$$(m_1\rho_1 + m_3\rho_3)\cos(\alpha_1 - c) - m_2\rho_2\cos(\alpha_2 - c) + m_1R_1\cos l_1 - m_2R_2\cos l_2 - m_3$$
$$(m_1\rho_1 + m_3\rho_3)\sin(\alpha_1 - c) - m_2\rho_2\sin(\alpha_2 - c) + m_1R_1\sin l_1 - m_2R_2\sin l_2 - m_3R_3\sin l_3$$
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In his paper on the "General Applicability of the Short Method" Leuschner draws the following conclusions about the indeterminate cases from a consideration of the relations existing among the differential corrections.

(1) Since the coefficient of $\partial\rho_0$ is independent of the solar coordinates the constants may be corrected differentially even if the body is in conjunction or opposition and near its node. In this case also a first approximation to the constants is derivable in a simpler manner than ordinarily.

(2) The coincidence of the first and third places does not necessarily make the solution indeterminate.

(3) When the observations lie in a great circle the solution becomes indeterminate only if three other conditions are fulfilled.

(a) When the intervals are equal.

(b) When the third powers of the intervals multiplied by the solar constant K are numerically ineffective.

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intervals, the coefficient of $\partial \rho_0$ becomes zero or nearly so.

Moulton draws the following conclusions in his paper in the Astronomical Journal:

When the determinant $\Delta_2 = \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix}$ vanishes, i.e. the three apparent positions lie on the arc of a great circle r_2 and ρ_2 can still be found if the quantities K and ϕ are large as compared with the neglected terms of higher order. An examination of these quantities shows that they are very small in the case where the sun lies on or near the same great circle so the result does not apply in this instance.

If the apparent positions of the observed body are the same at t_1 and t_3 the problem is in general not indeterminate, he states. Each of the determinants contains as a factor the sine of the angle between ρ_1 and ρ_3. After this factor is removed the determinants will not in general be zero and r_2 can be found as before. "In general it is not only not fatal if the three observed positions lie in a great circle but they may even be identical."

Andoyer (Bulletin Astronomique, 1917) comes to much the same conclusion. He states that since the directions of the sun are necessarily distinct there can only be an indetermination of the first order and that only in the case where these directions lie in the same plane as the single direction of the planet. So that even if all three directions coincided it would still be possible to have a solution.

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GREAT CIRCLE MOTION.

Bruns has shown that the determinant $\begin{vmatrix}\lambda & \lambda' & \lambda''\\ \mu & \mu' & \mu''\\ \nu & \nu' & \nu''\end{vmatrix} = V^3 k$ where k is the geodetic curvature of the apparent oribt on the sphere and V the velocity in this orbit at the point $(\lambda_1, \mu_1, \nu_1)$. From this we can see that the determinant can vanish if either:

(1) V = 0, the apparent geocentric velocity is zero, which corresponds either to the loop case or to motion in the line of sight,

or(2) k = 0, the apparent curvature is zero. This will be true if the body moves in a great circle or if there is a point of inflexion at the middle date.

Moulton has shown that it is impossible for a comet or planet to continue to move in any other great circle than the ecliptic and this, of course, only when the plane of the orbit coincides with the ecliptic. In such a case four observations are needed to make the problem determinate.

A point of inflexion evidently occurs when a comet's heliocentric distance changes from one greater than the earth's to one less (or vice versa) since then the geocentric path changes from convex to concave (or vice versa).

We have already seen that when $|\lambda\ \lambda'\ \lambda''| = 0$ one of two circumstances must also hold; either r = R or the great circle which touches the apparent orbit passes through the middle position of the sun. In practice, which of these is more likely to occur? Judging from the results obtained from using Leuschner's short method, it seems to be the former, for comets. When we

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examine a little closer we find that this is what would be expected. The cases exactly fulfilling the second set of conditions must of necessity be very few. Where the conditions are only approximately fulfilled, the auxiliary quantities used in the determination of ρ_2 may be small but being perfectly definite in value the problem can be solved without further difficulty.

In the case of minor planets it is obvious that since r can never equal R great circle motion is indeterminate in the case of elliptic motion unless more than three observations are used. With parabolic motion the problem can be solved in either event, either by the substitution of the value $r_2 = R_2$ or by a method analogous to the exceptional case in Olbers' method. Whatever holds true for the parabola where one element, the eccentricity, is given, also holds true when any other element is known. Therefore the case of a conditioned solution for an ellipse where we have a value for the period can be solved in instances where the ordinary elliptic problem becomes indeterminate.

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THREE OBSERVATIONS LIE ON A GREAT CIRCLE PASSING THROUGH THE MIDDLE PLACE OF THE SUN.

This means that the three geocentric lines of sight lie in a plane which also contains the line joining the sun and the earth at the date of the middle observation. Evidently, therefore, the earth is in or near the plane of the planet or comet's orbit at that date. In order that this condition should hold very exactly either the planet moves in the plane of the ecliptic or the time intervals are so short that the arc of the earth's orbit would be an infinitesimal from which the four lines of sight (three to planet, one to sun) radiate, all lying in one plane. If the earth were exactly in the plane of the orbit at the middle date the longitude of its ascending or descending node would be given by the longitude of the sun at that date and the inclination of the orbit could be readily derived from the values of the observed latitudes. In practice this would only be very approximately true as is evident from a consideration of the three exceptional cases cited by Weiss.

Name of Comet	L_2	Long. of Ascending Node	$i =$ arctg $\frac{tg\,\beta_2}{\sin(\lambda_2 - L_2)}$	Inclination
1869 III	108°	293°	20°	7°
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Cayley in his paper on the Geometrical Determination of the Orbit of a Planet from Three Observations states that the problem is only indeterminate if the plane of the orbit contains one of the three rays - or lines of sight. Since the intersection of the plane of the orbit with this ray is no longer

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fewer

The ordinary equations for determining ρ_2, whether in the Laplacian or in the Gaussian methods break down and assume the indeterminate form $\frac{0}{0}$. For example in the Gaussian method we have the equation:

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where

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Evidently in this case the determinants to all have identical columns and therefore vanish. In the Laplacian method if the middle date is at the turning point of the loop the velocities of the direction cosines must be zero and therefore this determinant must vanish

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THE FIRST AND THIRD GEOCENTRIC POSITIONS COINCIDE.

This means that as seen from the earth the planet or comet describes a loop or knot with the first and third positions coinciding at the nodal point, the middle observation lying somewhere on the loop itself. In order that the two positions shall appear to coincide it is evidently necessary that the lines of sight at the dates of the first and third observations should be parallel.

The ordinary equations for determining ρ_2, whether in the Laplacian or in the Gaussian methods break down and assume the indeterminate form $\frac{0}{0}$. For example in the Gaussian method we have the equation:

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The material that has been written on this case is conflicting. Gauss states that r_2 can be obtained but the positions of the first and third places remain arbitrary, subject only to the condition that they lie in the same plane as the second. Hansen and Bauschinger come to much the same conclusion.

On the other hand we have Weiss, Moulton, and Andoyer stating that the problem is not indeterminate. Weiss merely states that while r_2 is indeterminate the equations for ρ_1 a d ρ_3 still remain. According to Moulton the problem is indeterminate due to the fact that each determinant contains as a factor the sine of the angle between ρ_1 and ρ_3. This fact is also noted by Bauschinger. Moulton goes on to say that after this factor is removed the determinants will not in general be zero. Andoyer claims that even if all the directions were to coincide one could still obtain three equations among the geocentric distances and the observed quantities and finish as before.

A study of the equations given by these different writers fails to show how they can be solved under these circumstances. Unless some method not suggested in these papers is devised for transforming the equations there seems to be present an indetermination of the first order. That is, while we are able to obtain ρ_2 we can get ρ_1 and ρ_3 only in the combination .

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the conditions $\alpha' = \delta' = 0$ we have as follows

$$\sigma'' + \sigma\left(\frac{1}{r^3}\right) = \delta\left(\frac{1}{r^3} - \frac{1}{R^3}\right)\cos(A-\alpha)$$

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$$\sigma''\,\mathrm{tg}\,\delta + \sigma\left[(\mathrm{tg}\,\delta)'' + \frac{\mathrm{tg}\,\delta}{r^3}\right] = \delta\left(\frac{1}{r^3} - \frac{1}{R^3}\right)\mathrm{tg}\,D$$

We can find the value of σ from the second of these equations but σ' does not appear at all.

Professor Leuschner suggests a method of solution in this case by combination with the equations obtained in the other method. We have there:

$$m_1\rho_1 + m_3\rho_3 = c$$

We can also write by Taylor's series:

$$\rho_1 = \rho_2 + \theta_1\rho_2' + \frac{\theta_1^2}{2}\rho_2''$$

$$\rho_3 = \rho_2 + \theta_3\rho_2' + \frac{\theta_3^2}{2}\rho_2''$$

Substituting in these values we have

$$(m_1 + m_3)\rho_2 + (m_1\theta_1 + m_3\theta_3)\rho_2' + \left(m_1\frac{\theta_1^2}{2} + m_3\frac{\theta_3^2}{2}\right)\rho_2'' = c$$

which is an equation in the single unknown ρ since we can obtain values of ρ_2 and ρ_2'' from the usual equations.

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$$\rho_3 = \rho_2 + \theta_3\rho_2' + \frac{\theta_3^2}{2}\rho_2''$$

Substituting in these values we have

$$(m_1+m_3)\rho_2 + (m_1\theta_1 + m_3\theta_3)\rho_2' + \left(m_1\frac{\theta_1^2}{2} + m_3\frac{\theta_3^2}{2}\right)\rho_2'' = c$$

which is an equation in the single unknown ρ since we can obtain values of ρ_2' and ρ_2'' from the usual equations.

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CLASSIFICATION OF CASES FOR WHICH N BECOMES ZERO.

In Leuschner' short method the projected geocentric distance ($\sigma = \rho\cos\delta$) is given by the equation.

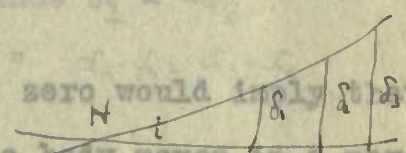
$$\sigma = -\frac{R\cos D\left[\alpha'(tg\,\delta\cos(A-\alpha) - tg\,D + (tg\,\delta)'\sin(A-\alpha)\right]}{\alpha'^3\, tg\,\delta - \alpha''(tg\,\delta)' + \alpha'(tg\,\delta)''}\left[\frac{1}{r^3} - \frac{1}{R^3}\right]$$

Let us now examine the cases where the denominator N vanishes and determine their geometrical meaning.

From analogy with other methods we should expect it to vanish as a whole if the three observations lie on a great circle. This can be very easily shown: In that case

$$tg\,\delta_2 = tg\,i\,\sin(\alpha_2 - H)$$

Differentiating and remembering that i and H are constants



$$(tg\,\delta)_2' = tg\,i\cos(\alpha_2 - H)\,\alpha_2'$$

$$(tg\,\delta)_2'' = tg\,i\left[\cos(\alpha - H)\,\alpha_2'' - \sin(\alpha_2 - H)\,\alpha_2'^2\right]$$

If now we substitute these values into N we have:

$$\alpha_2'^3\sin(\alpha_2 - H) - \alpha_2''\alpha_2'\cos(\alpha_2 - H) + \alpha_2'\alpha_2''\cos(\alpha_2 - H) - \alpha_2'^3\sin(\alpha_2 - H)$$

Therefore N vanishes when the three positions lie on a great circle.

Considering next the cases where N is zero because the separate terms are zero there are seen to be simply four cases.

(1) $\alpha' = \delta' = 0$

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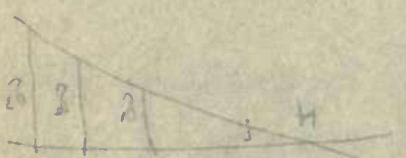

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Of course, under any one of these cases there will arise others where additional factors are also zero. These, however, will not be essentially different from the four types above but will be merely more complicated, involving additional conditions. They will be considered in a brief manner under the four cases.

CASE I $\alpha_0' = \delta_0' = 0$

This will mean that the geocentric path of the comet or planet has a turning point at the middle date, or has described a loop. If the time intervals are equal $\alpha_1 = \alpha_3$, $\delta_1 = \delta_3$ and the first and third positions coincide.

If in addition δ_0' is zero the loop is tangent to the equator at the middle date.

The additional condition δ_0'' is zero would imply that there was no motion in declination or the body moves in a parallel of declination, having a turning point at the middle date. This would seem an extremely unlikely occurrence - though theoretically possible.

CASE II $\alpha_0' = \alpha_0'' = 0$

The motion is in an hour circle and this is therefore a case of great circle motion.

If δ_0' is in addition zero there is a turning point at the middle date and the first and third positions nearly, if not exactly, coincide.

CASE III $\delta_0 = \delta_0' = \delta_0'' = 0$

In this case the motion is necessarily in the equator.

CASE IV $\alpha_0'' = \delta_0 = \delta_0'' = 0$

The apparent motion of the body is uniform, crossing the equator at the middle date. If the intervals are equal

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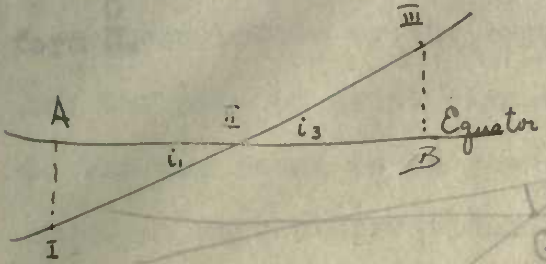
$$\alpha_0'' = 0 \quad \therefore \quad 2\alpha_2 = \alpha_1 + \alpha_3$$

also $\quad \delta_0 = \delta_0'' = 0 \quad \therefore \quad \delta_3 = -\delta_1$

For pass a great circle through the first and second positions, also one through the second and third positions. In order to prove that all three positions lie on the same great circle it is only necessary to prove that $i_1 = i_3$



Drop $\perp$s from I and III on the equator. The right spherical $\triangle$s IAII and IIBIII are equal.

IA = BIII since $\delta_1 = -\delta_3$

AII = IIB " $\alpha_2 - \alpha_1 = \alpha_3 - \alpha_2$

$i_1 = i_3$ and we have great circle motion.

We see that of these four cases where N vanishes one occurs when the geocentric motion forms a loop, the other three when the motion is in a great circle. In the former case we would expect that the numerator would also be zero.

LOOP CASE

The expression for the numerator is as follows:

$$-R\cos D\left\{\left[tg\,\delta_2\cos(A-\alpha_2) - tg\,D\right]\alpha_0' + \sin(A-\alpha_2)\,(tg\,\delta)_0'\right\}$$

Evidently this also vanishes when $\alpha_0' = \delta_0' = 0$ which was the condition that the planet or comet described a loop. We have then in this case the expression $\sigma = \frac{0}{0}$.

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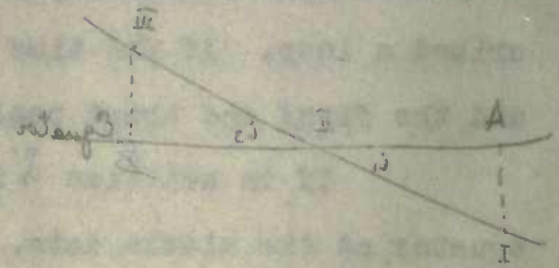

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We see that of these four cases where H vanishes one occurs when the geocentric motion forms a loop, the other three when the motion is in a great circle. In the former case we would expect that the numerator would also be zero.

<u>LOOP CASE</u>

The expression for the numerator is as follows:

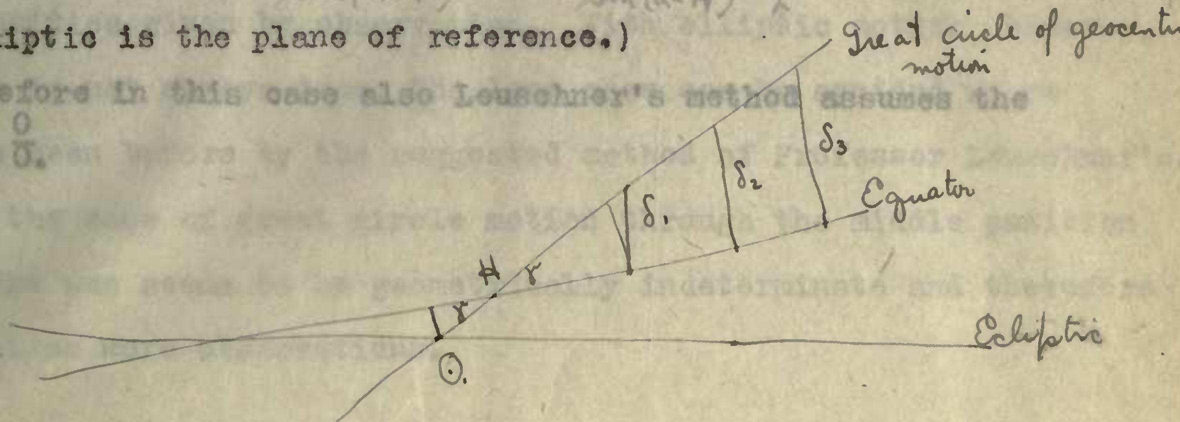
$$-R\cos P\left\{\left[\operatorname{tg}\delta_s\cos(A-\alpha_s) - \operatorname{tg}\delta\right]\alpha_0' + \sin(A-\alpha_s)(\operatorname{tg}\delta)_0'\right\}$$

Evidently this also vanishes when $\alpha_0' = \delta_0' = 0$ which was the condition that the planet or comet described a loop. We have then in this case the expression $\sigma = \frac{0}{0}$.

MOTION IN A GREAT CIRCLE PASSING THROUGH THE MIDDLE POSITION OF THE SUN.

It would be interesting to see whether in Leuschner's method also assumes the indeterminate form $\frac{0}{0}$.

Suppose the plane of reference is the plane of the equator. (It can be shown also more easily to hold when the ecliptic is the plane of reference.)



$$\operatorname{tg}\gamma \sin(\alpha_1 - H) = \operatorname{tg}\delta_1$$
$$\operatorname{tg}\gamma \sin(\alpha_2 - H) = \operatorname{tg}\delta_2$$
$$\operatorname{tg}\gamma \sin(\alpha_3 - H) = \operatorname{tg}\delta_3$$
$$\operatorname{tg}\gamma \sin(A_2 - H) = \operatorname{tg}D_2$$

Differentiating the second equation so as to find the velocities and accelerations at the middle date

$$\operatorname{tg}\gamma \cos(\alpha_2 - H)\,\alpha_2' = (\operatorname{tg}\delta_2)' ; \; -\operatorname{tg}\gamma \sin(\alpha - H)\,\alpha_1'^2 + \operatorname{tg}\gamma \cos(\alpha - H)\,\alpha_{12}'' = (\operatorname{tg}\delta)''$$

Substituting in the value of tan γ from the second equation

$$(\operatorname{tg}\delta)' = \frac{\operatorname{tg}\delta \cos(\alpha_2 - H)}{\sin(\alpha_2 - H)}$$

$$(\operatorname{tg}\delta)'' = \frac{\operatorname{tg}\delta}{\sin(\alpha_2 - H)}\left[-\alpha'^2 \sin(\alpha_2 - H) + \alpha'' \cos(\alpha_2 - H)\right]$$

If we now substitute this expression in the numerator and denominator of the value of σ.

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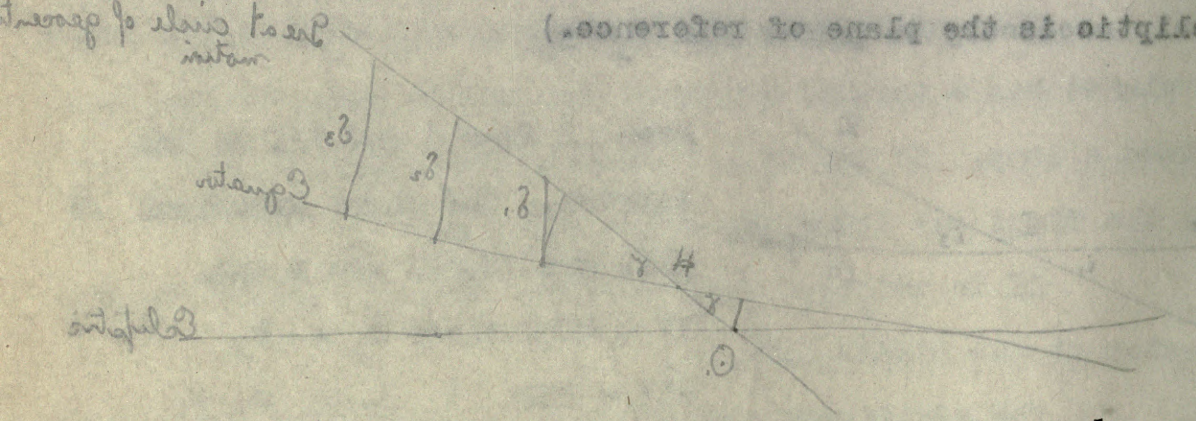

$$tg\,\gamma \sin(\alpha_1 - H) = tg\,\delta_1$$
$$tg\,\gamma \sin(\alpha_2 - H) = tg\,\delta_2$$
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If we now substitute this expression in the numerator and denominator of the value of G.

we have for the numerator

$$\alpha'\left\{tg\,\delta\cos(A-\alpha)-\frac{\tan\delta\sin(A-H)}{\sin(\alpha_2-H}\right\}+\frac{\sin(A-\alpha)\,tg\,\delta\cos(\alpha-H)}{\sin(\alpha-H)}\alpha'$$

$$=\alpha'\,tg\,\delta\left[\sin(\alpha-H)\cos(A-\alpha)-\sin(A-H)+\sin(A-\alpha)\cos(\alpha-H)\right)$$

$$=\alpha'\,tg\,\delta\left[\sin(A-H)-\sin(A-H)\right)=0$$

for the denominator

$$\alpha_1'^{3}\,tg\,\delta-\alpha_0''\alpha_1'\frac{tg\,\delta\cos(\alpha-H)}{\sin(\alpha-H)}+\alpha'\frac{tg\,\delta}{\sin(\alpha-H)}\left[-\alpha_0'^{2}\sin(\alpha-H)+\alpha_1''\cos(\alpha\right.$$

$$=0$$

Therefore in this case also Leuschner's method assumes the form $\frac{0}{0}$.

CONCLUSION.

The orbit problem in the case of parabolic motion then appears to be fully determinate under any of the circumstances discussed above. This is due to the fact that in that case we have five elements to be determined from the six quantities given by observation. With elliptic motion, however, this is not always true: The loop case can be avoided as we have seen before by the suggested method of Professor Leuschner's. But the case of great circle motion through the middle position of the sun seems to be geometrically indeterminate and therefore requires more observations.

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